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A NOTE ON STREAM ORDERING AND CONTOUR MAPPING

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34

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A NOTE ON STREAM ORDERING AND CONTOUR MAPPING

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This paper has been prepared in response to what might be regarded, in other arenas of endeavor, as "popular demand". We shall also use it to make a plea for certain changes in topographic mappings. In our first paper in the Harvard Papers in Theoretical Geography series, 16 May 1967, we examined surfaces in general terms but also recognized certain points, lines, and areas on surfaces as singular in their characteristics and worthy of specific delineation in that they represented that minimal part of the spatial structuring that need be known if the flows, if any, on surfaces were to be understood. A simple table was included that grouped the singular geometrical elements by dimensions and by vergency. It is included here subsequently. A hypothetical surface was presented to portray the ideas involved. Following Cayley (1859) and Maxwell (1870) we have discussed the topology and geometry of any surface and the relationship of the points, lines, and areas to flow phenomena. We have found that flows of energy and matter unite various parts of the surface into a system. Fluvial systems, for instance, are highly organized, and show systematic regularity as Horton (1945) first demonstrated, and as Strahler (1964) and his students have corroborated.

While useful for general "movement theory" purposes, and having been developed with social and economic applications in mind, our hypothetical surface did not contain sufficient "tree-like" or "bifurcation" properties to reveal, for example, the typical, indeed, the necessary, spatial features and their dimensional representation present in even moderately high order river basins. This present paper, then, is an attempt to clarify these matters and to accede to wishes that we examine river basin geometry explicitly.

None of the illustrations from the paper noted above will be reproduced here. The few following succinct definitions will, it is hoped, suffice.

A contour line may be regarded as the intersection of two surfaces, the conventional circumstance being that of the intersection of the variable surface under consideration and some specified constant-valued "level" surface. Any two intersecting surfaces do so along a line. This line thus connects points of equal value on the variable surface. Using conventional mathematical notions of surfaces, it is to be seen that the "z" value is constant along a contour line on a variable surface when "x" and "y" values are taken as coordinates on a referent level surface.

The summary of such values with relation to land forms phenomena are conveniently and convincingly displayed by means of a map on which selected contour lines are shown representing the intersections of level surfaces with the variable surface of the phenomenon and with each contour line distinguished by a numeral which shows the level surface to

which it belongs.

Let us now relate the nature and significance of contour lines to certain absolute extremum points (local maximum, or minimum) and mixed extrema points on a surface.

The contour "line" at a peak becomes a point. A peak is a local maximum of elevation. Everywhere in the immediate neighborhood on the surface elevation values are lower.

The contour "line" at a pit becomes a point. A pit is a local minimum of elevation. Everywhere in the immediate neighborhood on the surface elevation values are higher.

In general, of course, peaks are at higher elevations than pits. This is not inevitably required to be so for individual peaks or pits. The local condition of the surface determines which, if either, exists.

Peaks and Pits are to regarded as singular points and constitute the category we shall call absolute extremum points.

The other kind of singular points, i.e. mixed extrema points, are Passes and Pales. A pass or saddle point exists, for example, at the self-crossing point of some contour line that forms two loops, one around each of two adjacent peaks. To find the pass or passes relating to a given peak requires the establishment of the outermost closed contour line.

A pale exists between adjacent pits. The self-crossing contour line for adjacent pits may be either of the inloop or outloop type. The inloop type also consists of two closed curves, one of which, however, lies inside the other except for their shared point. Inloop types occur linking pits within streams and for lakes with outlets.

Within any given loop identified with a given singular point, other singular points and their attendant loops may be found. Through every point on the surface one may consider that there is not only a contour line, but also a slope line. Slope lines indicate the direction of steepest gradient at a given point and, of necessity, therefore, intersect the contour lines at right angles. We therefore have two families of orthogonal curves on the surface, namely contour lines and slope lines. The property of orthogonality is preserved in a conformal projection to the plane of these two systems.

In general, slope lines have peaks and pits as their termini. In general any one slope line leads to some peak in its "uphill" direction and to some pit in its "downhill" direction. If, however, a particular slope line is found to run from a pit to either a pass or pale, then when continued through that pass or pale, that line will, of necessity, run only to another pit, (or in rare cases, the same pit) provided, of course, that the surface be a continuous one as we have stipulated. Similarly any slope line found linking a peak to a pass or pale, must then, when extended through the pass or pale continue only to another peak (or in rare cases, the same peak).

Slope lines linking peaks via passes or pales are designated as Ridge lines. Slope lines linking pits via passes or pales are designated as Course lines. All slope lines other than course lines and ridge lines do not encounter passes or pales as they are traced between peaks and pits.

On continuous surfaces the point of the pale and of the pass have at once the attributes of both a local maximum and a local minimum, each resulting in a self-crossing contour. On a map showing the areal variation of the surface for any true field quantity, no flat areas are to be found. There are, of course, singular points on the surface where the instantaneous gradient goes to zero, and direction of slope becomes indeterminate when these points are regarded in isolation. These points are precisely of the kinds mentioned above, peaks, pits, passes, and pales. Peaks and pits have been defined as local maxima and local minima respectively; simple closed contours therefore exist within their immediate neighborhoods. At the point of a pale or of a pass, gradient is zero, but no locally closed contours occur in the immediate neighborhood; for the point of the pale or of the pass has the attributes both of a local maximum and of a local minimum, and a self-crossing of the contour line results. One profile, properly taken, shows the pass at the lowest value between two peaks; a profile through the same point taken at right angles to the first shows the value at the pass to be higher than any other along that cross section. The pale occurs at the high point between pits, but it is a low point along the line at right angles to the line joining the pits.

To make the above considerations clearer, let us employ the concept of the _indicatrix_, defined as the curve in which a given surface is cut by a plane indefinitely near and

parallel to the tangent plane at any point, so called because it indicates the nature of the surface at that point. In addition to contour lines, let lines of steepest slope also be considered and hereafter referred to simply as slope lines. Slope lines, of course, always cross contour lines at right angles. (This property is preserved on these conformal maps.) Through every point on the surface, there is a slope line that, in general, begins at a certain peak and ends at a certain pit. Exceptions are the lines connecting peaks and passes and those connecting pales and pits.

Consider the most distant closed contour line defining a given sample peak or pit. This exterior line is intersected at every one of its points by a slope line. All of these slope lines must intersect all of the interior closed contour lines as well, uniting at an interior point -- the peak or the pit. In the general case, the indicatrix at the peak or pit will be an ellipse, as will the immediate contour lines in the neighborhood, with the major and minor axes corresponding to the directions of least and greatest curvature respectively.

Letting a and b for the indicatrix be the semidiameters, major and minor respectively, the equation for the orthogonal trajectory of the ellipse on a plane with x and y coordinates is $y^{b^2} = Cx^{a^2}$. So long as C does not equal infinity, the curve this equation defines touches the axis of x, the direction of least curvature. If C does equal infinity, x becomes zero and the curve touches the axis of the direction of greatest curvature, i.e., y. In general, at the peak or

the pit all of the slope curves, save the one limiting case, touch the line indicating the direction of least curvature. The only exception occurs when the indicatrix is truly a circle and the slope lines pass in all directions through the common point at the peak or pit.

At a pale or a pass the indicatrix is, in general, a hyperbola and the trajectory is $C = x^{a^2} y^{b^2}$. When this goes through the pass, C equals zero, and then either x or y equals zero. As a result, only two slope lines occur through the pass, each bisecting the angles made by the branches of the self-crossing contour line.. These slope lines therefore intersect at right angles. Let the slope line on which the pass is a point of minimum elevation be termed a ridge line and the other, a course line.

In general, ridge lines pass from peak to peak and course lines from pit to pit. Even with an arbitrary boundary condition it is possible that any one ridge or course line begins and ends on a map at the same peak or pit, thus forming a closed curve, and, in exceptional cases, any one of these lines may be alternately a ridge line or a course line, as is the case with minor peaks on a major one, so that any particular segment must be named with reference to the pass or pale with which it is considered. This consideration must be regarded expecially in connection with the self-crossing contour line which is not of the usual figure eight or outloop type but rather of the inloop type.

To consider additional matters we need to establish symbols. Let:

S = number of peaks (summits),

I = number of pits (immits),

P = number of passes,

B = number of pales (bars),

C = number of course lines,

R = number of ridge lines.

Within any completely closed contour line on a continuous surface the number of peaks, $\underline{S}$, is always one more than the number of passes, $\underline{P}$, so that $\underline{S} = \underline{P} + 1$. The same rule applies to the number of pits, $\underline{I}$, and the number of pales, $\underline{B}$, so that $\underline{I} = \underline{B} + 1$.

If, in the singular cases of passes and pales, we count each of these as single, double, or n..ple. depending on whether two, three or $\underline{n}+1$ areas of elevation or depression meet at a pass or pale, respectively, then the above counts can be taken as before, giving each singlular point its proper number.

Let $\underline{P_1}$ be the number of single passes, $\underline{P_2}$ the number of double passes, etc., and $\underline{B_1}$, $\underline{B_2}$, etc., be the numbers of single, double, etc., pales. Then the number of peaks will be $S = 1 + P_1 + 2P_2 +$ etc., and the number of pits will be $I = 1 + B_1 + 2B_2 +$ etc.

Now, regard any one ridge line or course line as beginning at a pass or a pale and ending at its respective peak or pit. Then the number of ridge lines, $\underline{R}$, will be $\underline{R} = 2(B_1 + P_1) + 3(B_2 + P_2) +$ etc., and the number of course lines, $\underline{C}$, will be the same.

With reference to the area enclosed by any one contour

line, the following obtains: $(S + I) - (P + B) = 1$.

However, on a closed surface like the spherical earth: $(S + I) - (P + B) = 2$. For, on a sphere, a closed curve bounds two areas.

For a general topological consideration let $\underline{V}$ be the number of all singular points on the closed surface ($V = S + I + P + B$); let $\underline{E}$ be the number of lines ($E + R + C$); and let $\underline{F}$ be the number of separate faces or territories as we shall call them. Then, $F = E - V + 2$.

That is to say the number of faces plus the number of points minus the number of lines equal 2. Again, within any one closed contour line, $F = E - V + 1$.

This general topological relationship among points, lines, and areas was first established by Euler in network analysis (the bridge problem) and was explained in terms of any polyhedron where the number of faces plus the number of vertices minus the number of edges equals two.

Further consideration of the above shows that, for the world's surface, if we put E' equal to the number of ridge lines only, and V' equal to the number of peaks, passes, and pales, then F' is the number of districts of depression (dales) equal to the number of pits. If E'' specifies the number of course lines only and V'' the number of passes, pales, and pits, then F'' is the number of districts of elevation (hills) equal to the number of peaks when the two types of districts are taken independently. Districts whose lines of slope run to the same peaks are the hills, and those whose lines of

slope run to the same pits are the dales. The whole closed surface may be divided independently into hills and into dales, each point belonging to a certain hill and to a certain dale. Of course, ridge lines are the only slope lines not reaching pits, and course lines are the only slope lines not reaching peaks.

The table concerning flows referred to above is given below. We begin by assuming that "natural" movements on surfaces tend to be along steepest slope lines or gradient paths, i.e. at right angles to the contour lines, and from higher values on the surface toward lower values. These paths are minimum over-the-surface "distances" in each case. This elementary assumption is in keeping with the analysis of potentials and forces in general field quantity theory. Other assumptions about form and movement are possible especially with regard to additional forces besides the gradient force and also, long-run processes that change surfaces. Here, however, we restrict ourselves to simple gradient movements in the short-run. Our conclusions can be presented in a simple table summarizing converging and diverging flows in terms of dimensions.

DIMENSION	NAME OF SURFACE FEATURE	VERGENCY
Point	Peak	Divergence
	Pit	Convergence
	Pass	Mixed
	Pale	Mixed
Line	Course	Convergence
	Ridge	Divergence
Area	Hill	Divergence
	Dale	Convergence
	Territory	Mixed

Let us now relate these ideas specifically to fluvial systems on land forms, particularly to river basins. Figure 1 is an enlarged portion of the Belmont, N.Y., U.S. Geologic Survey Topographic map (contours only). The original linear scale was, of course, 1:62,500. The linear scale of the map shown here is about tripled (i.e., 1:20850). On the map in figure 1, one inch represents about one-third of a mile on the earth's surface. The contour interval is twenty feet.

It is often noted when instruction is offered concerning the making or interpreting of contour maps of physical land forms that contours crossing streams (we would say, contours crossing all course lines) are bent so that the "notches" point up stream, i.e., in the direction of higher elevations. We add that the contours crossing ridge lines are bent so that their notches point out precisely the down-hill direction. Even map readers of limited experience should have no trouble in picking out the most likely positions for streams and their accompanying basin divides on the above noted map in figure 1.

If we accept these patterns of contour bending as necessary -- and indeed they are necessary -- we can then follow through, to its final conclusion, the statement about the necessary elements and their dimensional relationships in fluvial systems.

On the following illustration, Figure 2, a simple first order stream (i.e. a stream having no tributary) and its basin area bounded by ridge lines are shown. The heaviest lines are ridge lines. The lines of intermediate width represent course lines. The solid part of it represents the portion lying in

Figure 1.

Enlarged Portion of the Belmont, N.Y., U.S.G.S. Topographic Survey Map (contours only)

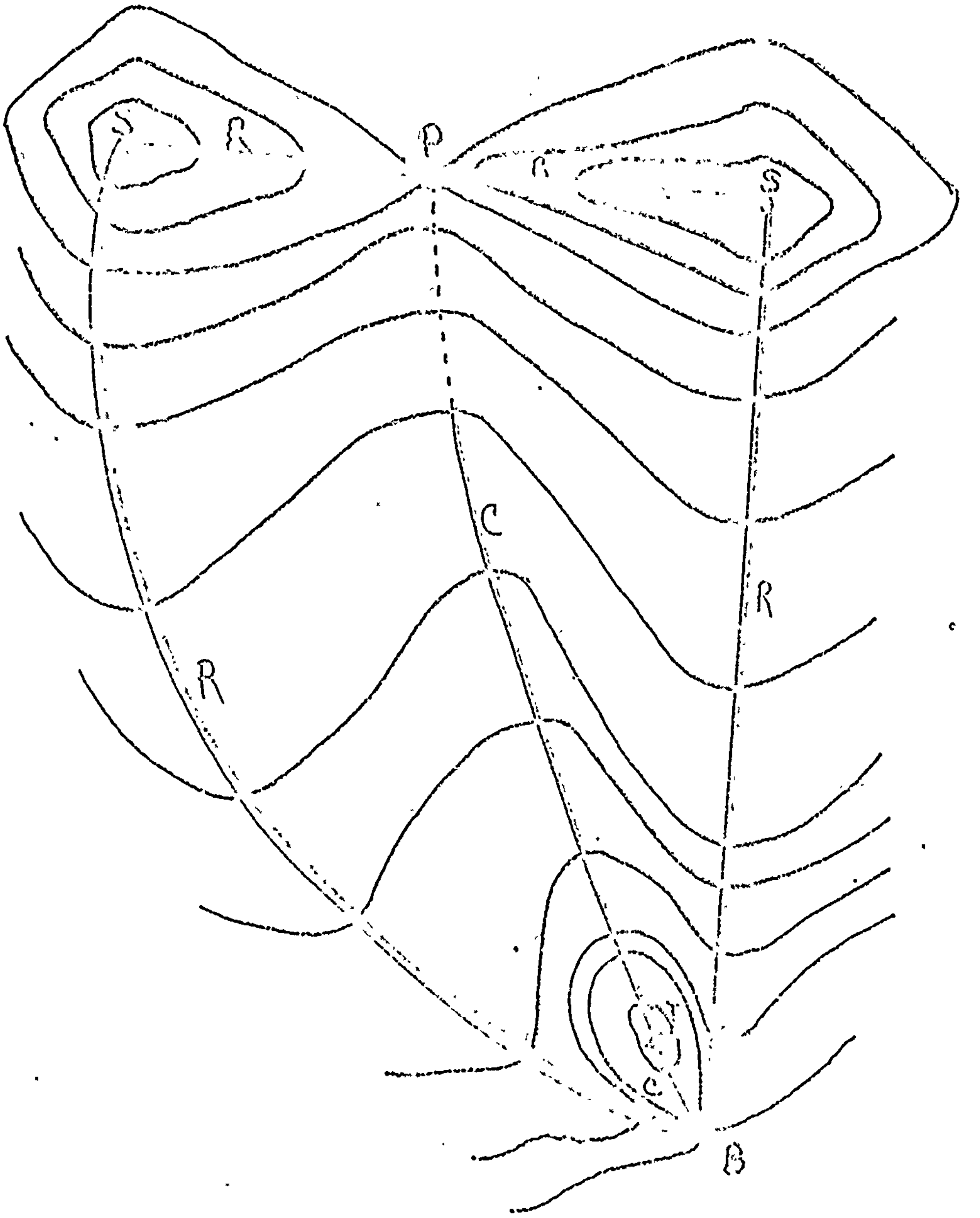

Figure 2.

First Order Basin

an actual stream and the dotted portion indicates the position of this line continued back to the pass whence it issued.

The fine thin lines are contour lines. Peaks, pits, passes and pales are labelled, as noted above, as S, I, P, and B respectively.

The significant features of the contours are the outloops of the figure of eight type contour with each loop enclosing a peak and with the self-crossing of the contour line occuring at the attendant pass and the inloop type bounding a pit with the self-crossing at the required pale. (Figure 3)

It is well established that any contour is a closed circuit on the earth (although, of course, not necessarily within the area portrayed on any given topographic map). With respect to a first order drainage basin the only contours that close within the basin are those in the immediate vicinity of the one pit. Moreover, all of these are within the smaller loop of the inloop type of contour (see figure 2, again). The larger loop lies entirely outside of the first order basin in question and its path may lead across many basins before closing. Note, however, that any one of the intermediate contour lines shown on figure 2 may itself be a segment of the larger loop of some inloop type contour with its smaller loop located entirely within some distant and separate basin. Even those contour lines that close in the immediate neighborhood of the peaks in figure 2 must lie only partly in the basin under consideration and of necessity partly outside it.

If other closed contours (say around some peak or some pit) are found within what has been previously assumed to be a first order basin, this is evidence that it is really some higher order basin and that a lower order one lies

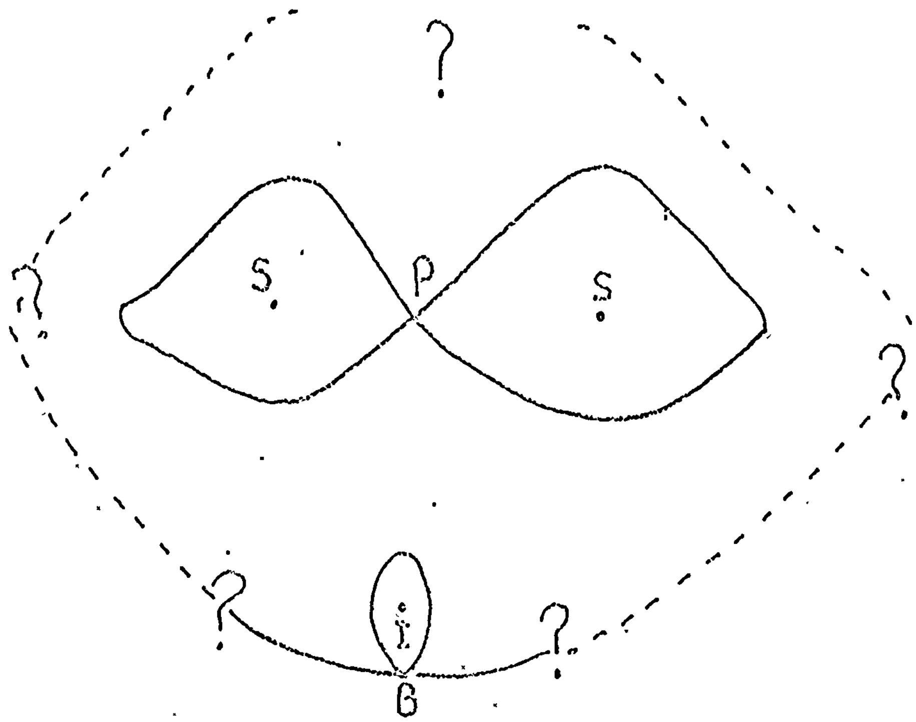

Figure 3.

entirely within it. In fact on the earth's surface a virtually infinite regress does exist, limited only by particle size at which level the concept of surface is not applicable. We shall speak of that difficulty later in this paper. For the present, we consider the presence of one visually observable unbranched stream -- having no tributaries -- as evidence of a first order stream (and its basin) and carry the accuracy of contour positioning to the level sufficient to delineate only it and not to include those evidences of additional course lines which, however, lack stream channel flows.

Note that course lines or ridge lines connect our designated points. Thus there are two individual and distinct course lines (segments) present within the basin. One runs downhill from the pass to the pit. The other continues from this pit uphill to the pale. For the stream to have a continuous flow there must be sufficient water in its channel to permit it a high enough level to cross the pale. Hence some water must flow uphill for all or any of it to move "downstream". Any stream flowing through several basins has a series of connected pits each with a "downstream" side but actually locally uphill pale. When streams dry up they do not do so in sheet-like fashion. Rather, they reduce to nothing through a series of disconnections of pits with resulting isolated pools of water. A disconnection occurs where and when there is not sufficient water for the flow to cross some pale.

Ridge lines also run between singular points, as for

example from a peak to a pass. Another ridge line then goes on to another peak. Also, ridge lines go from peaks to pales.

Ridge lines do not go to pits nor do course lines go to peaks. Both however do go to passes and to pales. If we consider peaks and pits as absolute extremum points and passes and pales as mixed extrema points, then any combination of course lines and/or ridge lines producing a connected continuous path through however many basins desired and of whatever order, do and must exhibit an alternation of absolute extremum and mixed extrema points. (This condition assumes additional importance when we examine below the infinite regress referred to previously.)

When two first order streams meet, a second order stream is formed. In figure 4, a typical example of the essential spatial structure attending the paired first order basins and the resulting second order stream is given.

Again we see that the outloop figure of eight type contour line is present, but this time there are two of them, one of them with both of its loops contained wholly within one of the loops of the other. The situation given in figure 5 below is, of course, possible, but not likely, the situation portrayed in figure 4 being the more general case. Figure 6-I shows a section of the inloop type of contour line we considered with the single first order basin. Figure 6-II shows us that a "rabbit-ears" or "butterfly" double smaller inloop type occurs with the incidence

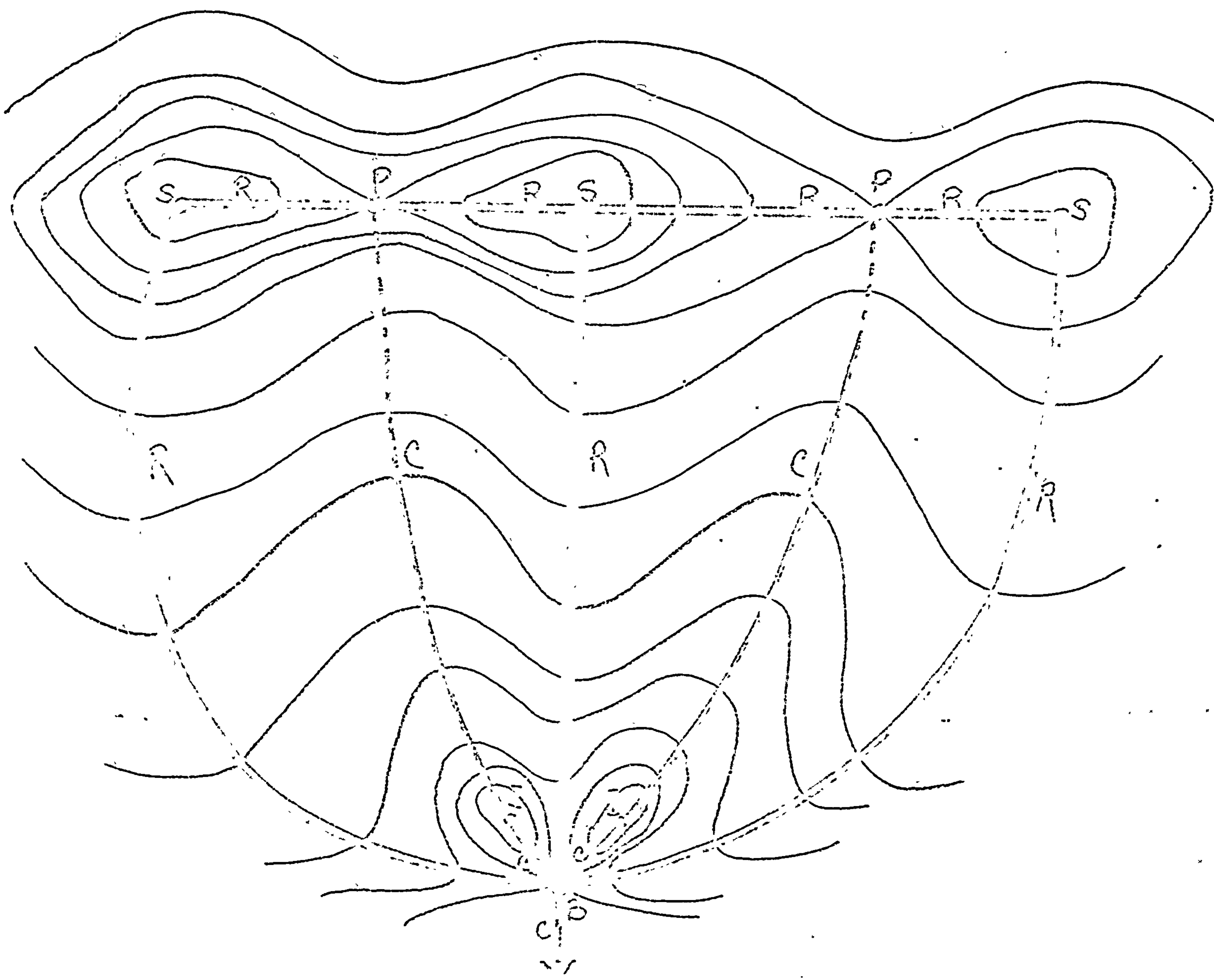

Figure 4.

Paired First Order Basins Producing Second Order Stream

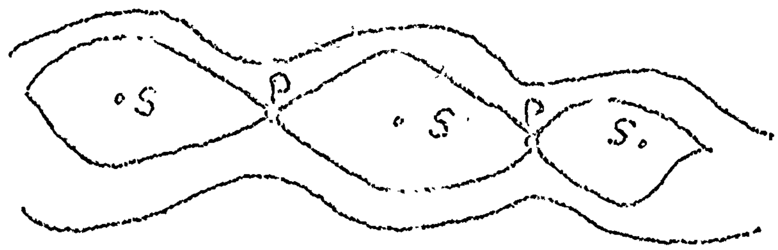

Figure 5.

Possible but Not Typical Arrangement
of Peaks and Passes

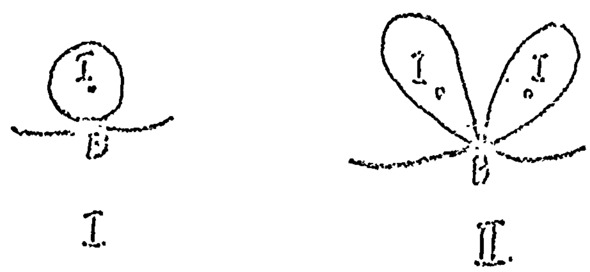

Figure 6.

Inloop Contours for Simple First
Order Stream (I) and
Paired First Order Streams (II)

of a second order stream.

Note on figure 4 that one ridge line (the "central" one) is shared as a common divide for the two first order basins.

Figure 7 reveals additional features concerning second order basins. This illustration is adapted from one by Strahler, op. cit., p. 376. He shows all streams within and the outer ridge line boundary of a fourth order basin. We have added plausible interior ridge lines and have marked singular points. Again, the heavy lines are ridge lines and the lighter ones are course lines. The second order stream is shown, as the legend indicates, by a dashed line. This illustration shows that the second order basin contains at least two first order basins whose exterior ridge lines it shares and additional area and ridge lines as well. Here the additional area is served by one additional first order basin and stream flowing into it, but not thereby promoting it to a higher order and by an area of direct drainage through no intermediating stream. Such an area of direct drainage (overland flow) into any stream of higher order than first is regarded as an interbasin area. We see then that two first order basins are a necessary but not sufficient condition for a second order basin. At least interbasin area must also exist. Of course additional first order streams not promoting the basin may develop within it, and indeed higher orders that do promote the basin may sometime come into existence.

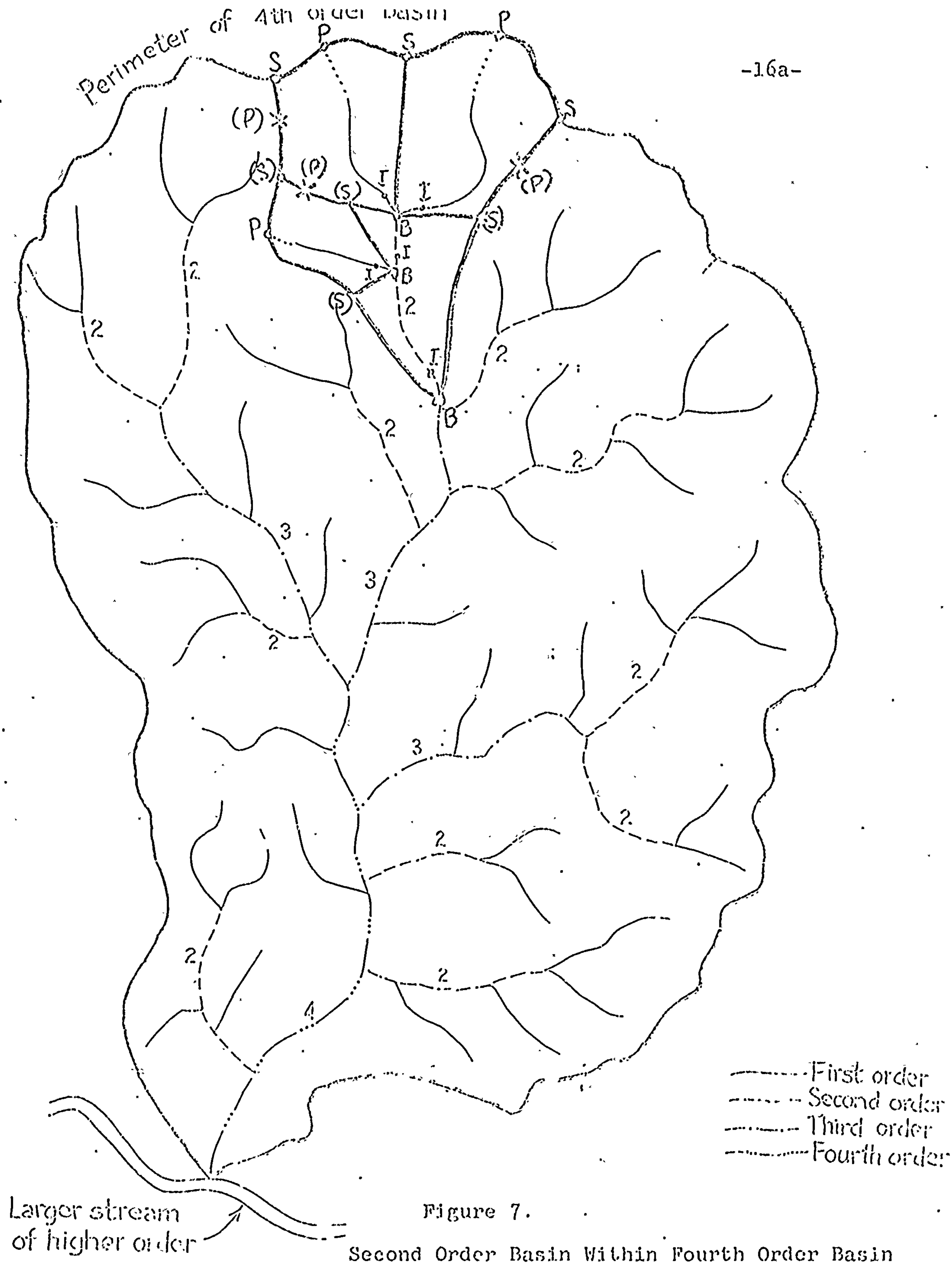

Figure 7.

Second Order Basin Within Fourth Order Basin

(Modified from Figure 25.7 in A.N. Strahler, Physical Geography, 2nd Ed., N.Y., 1960, p. 376.)

Always, however, <u>some</u> interbasin area <u>must</u> exist within <u>every</u> basin of order higher than one and very likely, based on observation, additional non-promoting lower order streams.

Now the other extremely important feature of figure 7 is that which shows that for basin orders higher than one additional course lines other than those for the first order streams <u>must</u> exist, albeit without channel flow. The x's located on ridge line segments indicate that additional heretofore unregarded passes (P) must exist on these segments. The precise locations of the x's must not be regarded as an attempt specifically to localize these passes but rather to indicate that somewhere on a ridge line segment between any two indicated peaks, S, and/or newly recognized peaks, (S), a pass must occur. (Recall the rule stated earlier concerning alternating absolute extremum and mixed extrema points along any connected path.) Now if these passes exist (and they must, as well as virtually innumberable others) so, too do their course lines, the resulting pits, and pales, and also the ridge lines that pales occasionalong with their additional peaks. These new peaks require additional passes, and so on in new and continually regenerating cycles to the level of separated particles at which level the concept of surface, itself, is no longer applicable. Obviously, the details of the entire structure of any set of nested "basins" (including those without channel flow) cannot be learned so that some threshold must be recognized and defined. It is apparent that all so-called recognizable first order streams do not contain precisely the same order course lines.

Actual rock type, climate, slope, etc., do determine the length of overland flow preceding channel flow. Moreover, "lower" orders do exist. However there is a demonstrated success that systematic relations among the observed empirical regularities in nested river basins can be understood by regarding as of first order the streams that have no visibly channeled flow tributaries. Despite variation in the actual course line order on a surface of so-called first order streams, the necessity to recognize a threshold exists and the regarding of all unbranched streams as of the same order, namely what has come to be called the first, has proved remarkably convenient and instructive operationally. The slightly different real order that each of the course lines of the so-called first order streams occupies in the hierarchy of course lines has not served to hide the nature of the system. Rather, the bold operational definition and ordering introduced from empirical observation has helped clarify the relationship in the system. We must remember however that our threshold thus defined has not so much a physical as a statistical meaning.

Figure 7 represents a plausible picture of a second order basin and indicates a likely basin arrangement based on flow efficiencies. It is, admittedly, possible to construct an entire hypothetical system for a high order basin in which a bifurcation ratio of two applies throughout and in which peaks are conserved. Thus, the regenerating cycle is avoided. However, either interbasin areas are too large and have

inefficient shapes or successively higher order stream segments then are shorter than preceding ones. For example, we can imagine a third order basin having only two second order streams and four first order streams. If peaks are to be conserved, then only five need exist, all on the exterior set of ridge line segments. Such a portrayal requires, contrary to observed regularities and "least-work" efficiency explanations of them, the inconsistencies noted above. Even a cursory glance at topographic survey maps confirms the notion of efficiency of size and shape of relevant areas and supports the necessity for and the existence of the virtually infinite regress among singular points, lines, and areas. Figure 7, then, presents a "reasonable" set of relations in that they are defensible by theory. Although virtually an infinite number of ridge lines exist, a tendency toward conservation of total ridge line length consistent with a given order of nested basins and work minimization principles seems to be the rule. That is to say, work minimization results in tendencies toward short ridge lines and compact basins. But note also sets of circular basins cannot exhaust an area.

Here then are the geometrical elements, according to their dimensions, present within or on the boundary of a first order basin, and the number of times each <u>does</u> <u>and</u> <u>must</u> occur. This is the geometry that is <u>necessary</u> <u>and</u> <u>sufficient</u> to a first order basin.

GEOMETRY OF FIRST ORDER BASIN

Element and Designation	Name and Designation	Number of occurrences (necessary and sufficient)
Point (V)	Peak (S)	2
	Pit (I)	1
	Pass (P)	1
	Pale (B)	1
Line (E)	Course (C)	2
	Ridge (R)	4
Area (F)	Dale	1
	Hill	0 (only parts of 2 separate hills are present)
	Territory	2

The geometrical elements represented in two first order basins paired to produce a second order stream (but not including the additional features of the second order basin) are given below.

GEOMETRY OF PAIRED FIRST ORDER BASINS

Element and Designation	Name and Designation	Number of occurrences (necessary and sufficient)
Point (V)	Peak (S)	3
	Pit (I)	2
	Pass (P)	2
	Pale (B)	1
Line (E)	Course (C)	4
	Ridge (R)	7
Area (F)	Dale	2
	Hill	0 (only parts of 3 separate hill are present)
	Territory	4

It is obvious that basin and dale are synonomous. The general principle that F + V-E = 1 and its variations including F' and F" as explained earlier not only holds within any one closed contour line, as it may or may not traverse several basins, but also within any one basin or paired basins. It is also equally obvious that one hill cannot be contained within a first order basin although single first order basins contain parts of two hills. It requires parts of four first or higher order basins or interbasin areas to comprise one hill. For example, in figure 4 the area within the course lines constitutes part of a hill, the other part lies on the other side of the external connected

ridge line segments between the two passes, which connected segments, in terms of topology, serve as an axis of symmetry.

The following relationships (see again Figure 2) hold for the single first order basin:

S = 2	R = 4	Dales = 1
I = 1	C = 2	Hills = 0 (parts of 2 separate hills)
P = 1		Territories = 2

$$F+V-E = 1$$
$$2+5-6 = 1$$

$$F'+V'-E' = 1$$
$$1+4-4 = 1$$

$$F''+V''-E'' = 1$$
$$0 +3 -2 = 1$$

The above conditions are those and only those which describe the geometry and topology of the single first order basin. They are at once necessary and sufficient and cannot be exceeded nor need they be in terms of efficiency.

For the paired first order basins (see Figure 4), we again find necessary and sufficient geometrical-topological conditions that obtain and that cannot nor need be exceeded for maximum efficiency:

S = 3	R = 7	Dales = 2
I = 2	C = 4	Hills = 0 (parts of 3 separate hills)
P = 2		Territories = 4
B = 1		

$$F+V-E = 1$$
$$4+8-11= 1$$

$$F'+V'-E' = 1$$
$$2 +6 -7 = 1$$

$$F''+V''-E'' = 1$$
$$0 +5 -4 = 1$$

Now, it is important to notice that the paired first order basins have an entity that permit them to be regarded as a unit. Their shared internal ridge line is an axis of symmetry. Woldenberg (1968) has demonstrated that mixed spatial hierarchies based on nested hexagons explain observed phenomena with great precision. It is therefore instructive to attempt to identify the basic hexagon in terms of our geometrical elements. What, for example, constitutes the one face of the modular hexagon? What are its vertices? --its edges?

Figure 8 is offered here as one possible identification. It does not, however, agree with the other details of Woldenberg's discovery and formulation. It is, therefore, only to be regarded as suggestive. In fact, nesting seems difficult. Other identifications are possible and, in fact, the recognition of course lines as edges seems to offer better possibilities in accounting for nestings.

Let us now depart from consideration of the identification of the hexagonal structure, if indeed, that identification is possible or, for that matter, desireable. Let us examine both the minimum second order basin and the typical second order basin. It is necessary to make this distinction for the minimum order geometrically cannot freely exist in face of work minimization and spatial efficiency. That is to say, a bifurcation ratio of two is not experienced in nature as the general case.

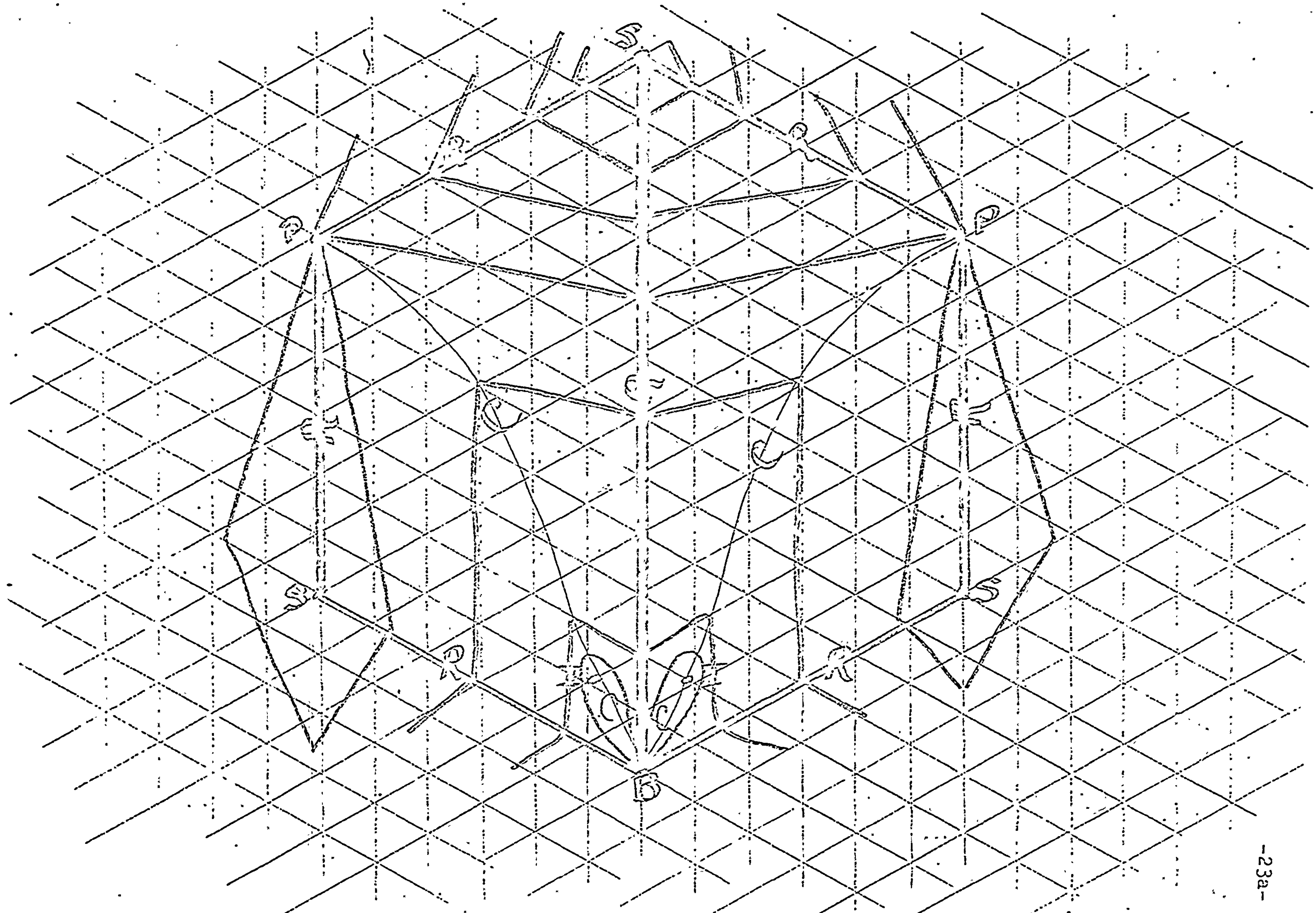

Figure 3.

Hypothetical Paired First-Order Basins-Hexagonal

Figure 9 shows the minimum second order basin. This condition is necessary, but not sufficient. For it, however,

S = 3	R = 9	Dales = 3
I = 3	C = 6	Hills = 0 (parts of 3 separate hills)
P = 2		Territories = 6
B = 2		

$$F+V-E = 1$$
$$6+10-15 = 1$$

$$F'+V'-E' = 1$$
$$3+7-9 = 1$$

$$F''+V''-E'' = 1$$
$$0+7-6 = 1$$

The structure in figure 9 conserves peaks, but is not efficient. A more reasonable one is that shown as noted, in figure 7. As noted earlier, the additional peaks require additional passes, and hence course lines, ridge lines, and thus pales and pits, and so on in regenerating cycles. It is important to note that the matching of the Euler theorem serves to identify various stages that can be regarded as consistent levels of generalization. The particular level to be shown is, of course, that one essential to the problem at hand.

These are important matters conceptually, in terms of the analysis of surface systems generally, and river basin systems in particular. They are also important cartographically. Cartographers through the ages have avoided showing the single-valued self-crossing contour lines. These are essential. Surveys should especially determine their precise locations

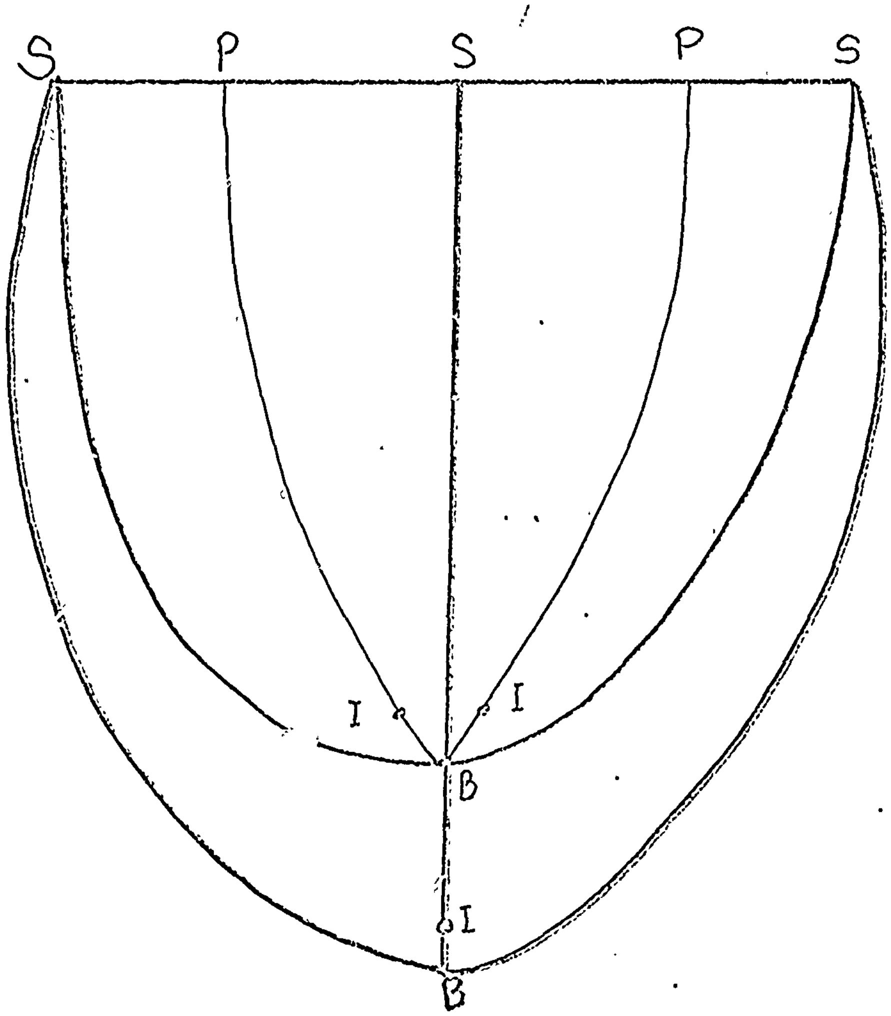

Figure 9.

Absolute Minimum Second Order Basin

and mapping should, above all else, indicate them, guided by the level of generalization needed.

Conventional topographic mapping practice has not reflected the developments in the understandings of the topology and geometry of surfaces and the study of flows on these surfaces. There is a need now for a new kind of topographic map which will be keyed to more recent knowledge in these fields.

The following paragraphs serve not only to recapitulate the foregoing arguments, but also to recommend new departures in topographic mappings. My colleague, Michael Woldenberg, has assisted in the preparation of these statements.

Specifically, current topographic maps generally use a constant contour interval to indicate gradients on surfaces. This practice is sufficient to describe flows on surfaces in general, if flow were simply overland flow, but of course this is only part of the story. Flows concentrate in channels to gain economies of scale, and hence the end points of course lines, along whose length channel flow begins and ends, must be clearly delineated. Thus a pass is a point and must be defined where some given contour line crosses itself. The pass is also located at the intersection of a course line and ridge line. As the ridge line is followed toward a confluence, it is found to recross the course line at its lower end, and a new point is located, the pale, which again may alternatively be described as the intersection point of a self-crossing contour line.

Thus a whole set of contours is called for, which will

serve to delineate these critical passes and pales. Between these contours normal interpolations may be used if desired.

The order reflected in Horton's laws may be reflected in the contour intervals. Suppose that in a river system in a region of youth, developed on a homogeneous surface, detritus is supplied so quickly that materials in the riverbed are not appreciably decreased in size with distance travelled downstream. Then Hack (1957) and Broscoe (1959) suggest the profile of the stream will be logarithmic, and hence the contour interval may well show arithmetic increase, where each contour interval reflects a new stream order.

Suppose the materials in the river bed decline exponentially with distance travelled downstream. Then the profile will be a power function, and the appropriate contour interval will be logarithmic, where each contour interval will correspond to the drop a stream makes for each order basin.

In a word we propose to perform research in developing new topographic maps reflecting the current understanding of spatial order on the earth's surface.

In addition, we propose a small change in the stream ordering system currently employed. In the preceding sections the convenient and operationally significant system employed by Horton as modified by Strahler was used. Basically, we suggest merely that what are now regarded as first order streams and basins be considered as of zero order and that present second order be called first and so on. The justification

for and the convenience attending such a change are described below.

Horton (1945) suggested several laws of fluvial morphology. These may be classified in two ways, direct geometric series, and inverse geometric series. Examples of these follow.

Direct geometric series:

In the law of stream lengths, Horton (1945, p. 291) stated that, "The average lengths of stream of each of the different orders in a drainage basin tend closely to approximate a direct geometric series in which the first term is the average length of streams of the first order."

The mathematically equivalent statement is:

$$\bar{L}_u = \bar{L}_1 R_L^{u-1}$$

where $\bar{L}_u$ is the average length of the stream of order u.
L_1 is the average length of the stream of order 1.
R_L is the length ratio of $\frac{L_{u+1}}{L_u}$.

Schumm (1956, p. 606), following the suggestion of Horton (1945, p. 294), created an analogous law for stream basin areas:

$$\bar{A}_u = \bar{A}_1 R_a^{u-1}$$

where $\bar{A}_u$ is the mean area of basins of order u, $\bar{A}_1$ is the mean area of the first order basins, and R_a is an area ratio analogous to the length of ratio R_L.

Inverse geometric series:

Horton's law of stream numbers is as follows (Horton, 1945, p. 291):

> "The numbers of streams of different orders in a given drainage basin tend closely to approximate an inverse geometric series in which the first term is unity and the ratio is the bifurcation ratio."

In the symbols

$$N_u = R_b{}^{k-u}$$

where N_u is the number of streams of order u, R_b is the bifurcation ratio N_u/N_{u+1}, and k is the order of the trunk stream.

As Woldenberg (1966, p. 433) has shown, stream order is a logarithm. Logarithms are by definition exponents to some base. Ten is commonly used, and an increase in order of magnitude is equal to 10^{x+1} (where x is any number).

Thus stream order is really stream order of magnitude, to the base of some ratio which may well differ from the commonly used 10. For direct geometric series, the variable y which is the function of stream order may be generalized as:

$$\bar{y}_u = y_1 (R_y)^{u-1}$$

where all terms are analogous to those previously identified.

If the threshold stream were actually of order 0 rather than order 1, this statement could be rewritten as:

$$\bar{y}_u = y_0(R_y)^u$$

Similarly inverse geometric series would be written as follows:

$$y_u = R_b^{k-u}$$

and would be unchanged by the change in the order for threshold streams.

Hence identifying the threshold stream as order 0 simplifies direct geometric series, and has no effect on inverse geometric series. There is another justification, as well.

We have pointed out above, the existence of the virtually infinite regress of peaks, pits, passes, pales, course lines, and ridge lines, basins, and hills in regenerating cycles which are implied, but not perceived on a topographic surface. Such course lines do not form channels, and such basins are thus not of threshold size. Therefore, when considering these basins, negative orders may well be used. It is possible to think of order equal to -5 or -19, etc., which signify basins at very small orders of magnitude.

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13. ABSTRACT

This paper describes the necessary, sufficient, possible, and likely geometrical and topological characteristics of stream drainage basins of various orders. A suggestion is made for experimental topographic mapping to bring it closer to the current theory of spatial hierarchical systems. Stream ordering is examined and a recommendation is offered that streams now considered as of first order be regarded as of zero order. This simplifies mathematical notation for the Horton-Strahler-Woldenberg type of analysis in quantitative geomorphology and has added intellectual interest by facilitating the understanding of the virtually infinite cyclical regress in the patterns of surface features down to individual particle size.